2023-24 MATH2048: Honours Linear Algebra II

Homework 2

Due: 2023-09-22 (Friday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date.

1. Let V and W be vector spaces over the field F, and let V_1 and W_1 be subsets of V and W respectively. Consider the direct product $V \times W$.

   Prove or disprove: If V_1 is a subspace of V and W_1 is a subspace of W, then $V_1 \times W_1$ is a subspace of $V \times W$.

   Prove or disprove: If the product set $V_1 \times W_1$ is a subspace of $V \times W$, then V_1 is a subspace of V and W_1 is a subspace of W.

2. Let V be a finite dimensional vector space and W be a subspace of V. Define a map $\pi : V \to V/W$ by $\pi(v) = v + W$ for all $v \in V$. Show that π is a surjective linear transformation and its kernel is W.

3. Let $\{v_i\}_{i\in I}$ be a spanning set of a (maybe infinite-dimensional) vector space V. Prove that there exists a subset $S \subseteq I$ such that $\{v_i\}_{i\in S}$ is a basis of V. (Hint: Use Zorn's lemma to prove a maximal S exists.)

4. (2.1 Q20) Let V and W be vector spaces with subspaces V_1 and W_1, respectively. If $T : V \to W$ is linear, prove that $T(V_1)$ is a subspace of W and that $\{x \in V : T(x) \in W_1\}$ is a subspace of V.

5. (2.1 Q13) Let V and W be vector spaces, let $T : V \to W$ be linear, and let $\{w_1, w_2, ..., w_k\}$ be a linearly independent subset of $R(T)$. Prove that if $S = \{v_1, v_2, ..., v_k\}$ is chosen so that $T(v_i) = w_i$ for $i = 1, 2, ..., k$, then S is linearly independent.

The following are extra recommended exercises not included in homework..

1. Summarize the concepts met so far. Make a mindmap of the concepts, definitions, examples and theorems.

2. Read Artin Algebra, Second Edition, Section 3.6 (2 pages), or any other paragraphs about direct sums.

3. Let W_1 be the space of $n \times n$ matrices over $\mathbb{R}$ whose trace is zero. The trace of a square matrix is defined as the sum of its diagonal entries. Find a subspace W_2 such that $\mathbb{R}^{n\times n} = W_1 \oplus W_2$. (Hint: consider the nature of matrices in W_1 and what kind of matrices are not in W_1.)

4. (2.1 Q16) Let $T : P(\mathbb{R}) \to P(\mathbb{R})$ be defined by $T(f(x)) = f'(x)$. Recall that T is linear. Prove that T is onto, but not one-to-one.